10. Strategic Interaction

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MRes Microconomics

Overview

Before: Choice theory. Individual choice, one DM.

Now: Game theory. Multiple agents.

- Penalty kicker shoots left or right; model their behaviour as maximising prob. of scoring a goal.
- Goal-keeper goes left or right; model their behaviour as maximising prob. preventing a goal from being scored.
- Whether goal is scored or not depends on both their actions.

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Goal: understand mechanisms, rationalise behaviour, make predictions.

- What if the kicker is better with the left foot?
- Would the goalkeeper have done their research on the opponent?
- Is it a high stakes game?
- How does it depend on experience? What if the wind/sun/etc. is going in a particular way?

Overview

Applications abound:

- Investment decisions: buy/not buy stock; value of stock depends on others' decisions; speculative attacks.
- Politics: designing voting rules and the agenda.
- Firm competition and industrial organisation: pricing strategies by firms are analysed by game theoretic models to determine collusion.
- Auction theory (branch of game theory): spectrum auctions.
- Public economics: procurement policies.
- Evolutionary game theory: cancer treatment research.
- School choice: students choose strategically; other students' choices affect their outcome.
- Organisational economics: delegation of decision power within a firm or organisation.
- Education economics: outcomes and degree of competition in grading schemes.
- $\vdots$

Overview

1. Strategic Interaction

2. Normal-Form Games

3. Strict Dominance

4. Iterated Elimination of Strictly Dominated Strategies (IESDS)

5. Weak Dominance

6. Rationalisability

7. Level-*k*

8. More

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Normal-Form Games

A **normal-form game** is a tuple $\Gamma = \langle I, S, u \rangle$ where

- **Set of Players:** $i \in I$.
- **Strategy Space:** $s_i \in S_i$
- Strategy profile: $s \in S := \times_{i \in I} S_i$; $s_{-i} \in S_{-i} := \times_{j \in I: j \neq i} S_j$.
- **Payoff Function:** $u = \{u_i, i \in I\}$, $u_i : S \to \mathbb{R}$.

Interpretation: players have preferences over outcomes and each strategy profile s pins down an outcome (potentially the same outcome).

More on this later with extensive-form games.

Write $u_i(s) = u_i(s_i, s_{-i})$.

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Y is **mutual knowledge** = all players know Y

Y is **common knowledge** = all players know Y, all players know that all players know Y, all players know that all players know that all players know Y, etc.

Game of complete information: all aspects of the game are common knowledge.

Assume that all games are of complete information; later we'll discuss games of incomplete information.

Normal-Form Games

Strategies

- **Pure strategy** $s_i \in S_i$.
- **Mixed strategy** $\boldsymbol{\sigma}_i \in \Sigma_i := \Delta(S_i)$; $\boldsymbol{\sigma} \in \Sigma := \times_{i \in I} \Delta(S_i)$; $\boldsymbol{\sigma}_{-i} \in \Sigma_{-i} := \times_{j \in I : j \neq i} \Delta(S_j)$.

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- **ATT!** $\Sigma := \times_{i \in I} \Delta(S_i) \neq \Delta(\times_{i \in I} S_i)$. Why? Example?

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- **ATT!** $\Sigma := \times_{i \in I} \Delta(S_i) \neq \Delta(\times_{i \in I} S_i)$. Why? Example?
- Write $\boldsymbol{\sigma}(s)$ for $\prod_{i \in I} \boldsymbol{\sigma}_i(s_i)$.
- **Expected payoff** $u_i : \Sigma \to \mathbb{R}$ (slight abuse of notation)
- $u_i(\boldsymbol{\sigma}) := \mathbb{E}_{\boldsymbol{\sigma}}[u_i] = \sum_{s \in S} \boldsymbol{\sigma}(s) u_i(s) = \sum_{s \in S} \prod_{j \in I} \boldsymbol{\sigma}_j(s_j) u(s)$.

Interpretation: u_i as Bernoulli index; players EU maximisers.

Overview

Solution Concepts

Solution concept: Takes game Γ and makes predictions regarding outcomes.

Singleton-valued $\Gamma \mapsto S$.

Set-valued (what can and cannot happen) $\Gamma \mapsto 2^S$.
(Different from multiplicity.)

Deterministic vs Stochastic prediction: considering S or Σ or $\Delta(S)$.

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Existence: something is predicted.

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Continuity of the prediction?

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For simplicity, assume game is finite, $|S| < \infty$.

Results generalise beyond finite games, but require some care in definitions and, sometimes, restrictions on S_i and u_i (e.g., compactness, continuity, etc.).

Modified Split or Steal (Golden Balls, ITV 2007-09)

		Col Player	
		Split	Steal
Row Player	Split	J/2, J/2	0, J
	Steal	J, 0	J/4, J/4

Players? Strategies?

Payoffs?

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Players? Strategies?

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Prediction?

Strict Dominance

Definition

Fix $\Gamma = \langle I, S, u \rangle$.

(i) Strategy $\sigma_i \in \Sigma_i$ of player i **strictly dominates** strategy $\sigma_i' \in \Sigma_i$ iff $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$ $\forall \sigma_{-i} \in \Sigma_{-i}$.

(ii) Strategy $\sigma_i \in \Sigma_i$ of player i is **strictly dominant** iff it strictly dominates every $\sigma_i' \in \Sigma_i \setminus \{\sigma_i\}$.

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Idea: strong predictions

No one chooses strictly dominated strategies as there is something else that is strictly better.

If a strategy is strictly dominant, all others are strictly dominated, the player better choose the strictly dominant one.

Strict dominance is ordinal concept: doesn't matter if dominates by a little or a lot.

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Note: Dominance relation between strategies $\neq$ Pareto dominance of outcomes

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Is it reflexive? Complete? Transitive? Does it induce a lattice?

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Lemma

There can be at most one strictly dominant strategy for each player.

(Why?)

Strictly Dominant Strategy

Lemma

If $\boldsymbol{\sigma}_i$ is strictly dominant, then $\exists s_i \in S_i : \boldsymbol{\sigma}_i(s_i) = \mathbf{1}$.

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Proof

Suppose not. Then $u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) > u_i(s_i, \boldsymbol{\sigma}_{-i}) \forall s_i \in \text{supp}(\boldsymbol{\sigma}_i)$.

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But this implies that $u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i} \sigma_i(s_i) u_i(\sigma_i, \sigma_{-i}) > \sum_{s_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) = u_i(\sigma_i, \sigma_{-i})$, a contradiction. $\square$

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In other words, only pure strategies are strictly dominant.

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Proof

$\Longrightarrow$: By definition.

$\Longleftarrow$:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i \setminus \{s_i\}, s_{-i} \in S_{-i}$$

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Yes, it suffices to check opponents' pure strategies to assess if strategy strictly dominated.

Often say σ_i is strictly dominated (you'll need to explain what strictly dominates σ_i).

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Proof

$\Longrightarrow$: By definition.

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σ_i is strictly dominated by σ_i' $\Longrightarrow$ $u_i(\sigma_i, s_{-i}) < u_i(\sigma_i', s_{-i})\ \forall s_{-i} \in S_{-i}$

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Strictly Dominated Strategy

		Col Player	
		L	R
	T	0,0	3,2
Row Player	M	1,4	1,1
	B	3,0	0,1

Which strategy is strictly dominated?

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No pure strategy of Player 1 strictly dominates another pure strategy.

However: 1/2 T + 1/2 B strictly dominates M!

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However: 1/2 T + 1/2 B strictly dominates M!

Moral of the story: you may need to consider mixed strategies to assess which strategies are strictly dominated

I.e., it suffices to check *opponents' pure strategies* to assess if strategy strictly dominated, but do *need to check own mixed strategies*.

Strictly Dominated Strategy

If mixed strategy is stricty dominated, is there a pure strategy which is strictly dominated?

Not necessarily...

		Col Player	
		L	R
	T	-4,0	3,2
Row Player	M	1,1	1,1
	B	3,4	-4,1

P1 has no strictly dominated pure strategy, but 1/2 T + 1/2 B is strictly dominated by M.

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Lemma

If s_i is strictly dominated, then any $\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0$ is strictly dominated.

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Define $\sigma_i'' : \sigma_i''(s_i') := \sigma_i(s_i)\sigma_i'(s_i') + 1\{s_i' \neq s_i\}\sigma_i(s_i')$. WTS $\sigma_i'' \in \Sigma_i$.

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(i) $\sigma_i'' \geq 0$ and (ii) $\sum_{s_i'} \sigma_i''(s_i') = \sigma_i(s_i) \sum_{s_i'} \sigma_i'(s_i') + \sum_{s_i' \neq s_i} \sigma_i(s_i') = \sigma_i(s_i) + 1 - \sigma_i(s_i) = 1$.

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$\exists \boldsymbol{\sigma}_i' : s_i$ strictly dominated by $\boldsymbol{\sigma}_i'$.

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Then, $\forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i}$,

$$u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) = \boldsymbol{\sigma}_i(s_i)u(s_i, \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i')u_i(s_i', \boldsymbol{\sigma}_{-i})$$

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$$\begin{aligned} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) &= \boldsymbol{\sigma}_i(s_i)u(s_i, \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i')u_i(s_i', \boldsymbol{\sigma}_{-i}) \\ &< \boldsymbol{\sigma}_i(s_i)u(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i')u_i(s_i', \boldsymbol{\sigma}_{-i}) \end{aligned}$$

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u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) &= \boldsymbol{\sigma}_i(s_i) u(s_i, \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i') u_i(s_i', \boldsymbol{\sigma}_{-i}) \\
&< \boldsymbol{\sigma}_i(s_i) u(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i') u_i(s_i', \boldsymbol{\sigma}_{-i}) \\
&= \sum_{s_i'} \left[\boldsymbol{\sigma}_i(s_i)\boldsymbol{\sigma}_i'(s_i') + \mathbf{1}\{s_i' \neq s_i\}\boldsymbol{\sigma}_i(s_i')\right] u_i(s_i', \boldsymbol{\sigma}_{-i})
\end{aligned}
$$

Strictly Dominated Strategy

Lemma

If s_i is strictly dominated, then any $\boldsymbol{\sigma}_i \in \Sigma_i : \boldsymbol{\sigma}_i(s_i) > \mathbf{0}$ is strictly dominated.

Proof

$\exists \boldsymbol{\sigma}_i' : s_i$ strictly dominated by $\boldsymbol{\sigma}_i'$.

Define $\boldsymbol{\sigma}_i'' : \boldsymbol{\sigma}_i''(s_i') := \boldsymbol{\sigma}_i(s_i)\boldsymbol{\sigma}_i'(s_i') + \mathbf{1}\{s_i' \neq s_i\}\boldsymbol{\sigma}_i(s_i')$. WTS $\boldsymbol{\sigma}_i'' \in \Sigma_i$.

(i) $\boldsymbol{\sigma}_i'' \geq \mathbf{0}$ and (ii) $\sum_{s_i'} \boldsymbol{\sigma}_i''(s_i') = \boldsymbol{\sigma}_i(s_i) \sum_{s_i'} \boldsymbol{\sigma}_i'(s_i') + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i') = \boldsymbol{\sigma}_i(s_i) + \mathbf{1} - \boldsymbol{\sigma}_i(s_i) = \mathbf{1}$.
Then, $\forall \boldsymbol{\sigma}_{-i} \in \Sigma_{-i}$,

$$
\begin{aligned}
u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) &= \boldsymbol{\sigma}_i(s_i) u(s_i, \boldsymbol{\sigma}_{-i}) + \sum_{s_i' \neq s_i} \boldsymbol{\sigma}_i(s_i') u_i(s_i', \boldsymbol{\sigma}_{-i}) \\
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&= \sum_{s_i'} \left[\boldsymbol{\sigma}_i(s_i)\boldsymbol{\sigma}_i'(s_i') + \mathbf{1}\{s_i' \neq s_i\}\boldsymbol{\sigma}_i(s_i')\right] u_i(s_i', \boldsymbol{\sigma}_{-i}) = u_i(\boldsymbol{\sigma}_i'', \boldsymbol{\sigma}_{-i})
\end{aligned}
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□

Strictly Dominated Strategy

Lemma

If s_i is strictly dominated, then any $\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0$ is strictly dominated.

A mixed strategy involving a strictly dominated strategy is strictly dominated.

Strictly Dominated Strategy

Lemma

If s_i is strictly dominated, then any $\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0$ is strictly dominated.

A mixed strategy involving a strictly dominated strategy is strictly dominated.

Can also show the more general but arguably less useful property:

Lemma

σ_i is strictly dominated $\iff \forall \alpha \in (0,1], \forall \sigma_i' \in \Sigma_i, \alpha\sigma_i + (1-\alpha)\sigma_i'$ is strictly dominated.

Overview

Iterated Elimination of Strictly Dominated Strategies (IESDS)

Motivation: 'Common knowledge of rationality' (CKR)

CK that players maximise payoffs.

Payoff maximisation = means to describe behaviour $\implies$ CKR = CK of how people behave.

Know strictly dominated strategies not chosen. Know that everyone knows that strictly dominated strategies not chosen $\implies$ can ignore strictly dominated strategies.

Iterate reasoning...

Iterated Elimination of Strictly Dominated Strategies (IESDS)

Definition

Given $\langle I, S, u\rangle$, $S^\infty \subset S$ survives IESDS iff $S^\infty = \times_{i\in I} S_i^\infty$ and $\exists (S_i^k)_{k\geq 0}$ s.t.

(i) $S_i^0 := S_i$ and $S_i^\infty = \cap_{k\geq 0} S_i^k$;

(ii) for $k \geq 1$, $S_i^k \subseteq S_i^{k-1}$;

(iii) for $k \geq 1$, $s_i \in S_i^{k-1} \setminus S_i^k$ is strictly dominated in the restricted game $\langle I, \times_j S_j^{k-1}, u\rangle$;

(iv) No $s_i \in S_i^\infty$ is strictly dominated in the game $\langle I, S^\infty, u\rangle$.

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Remark

In finite games ($|S| < \infty$) order of elimination doesn't matter: always get the same limit set S^∞.

Beyond finite games, sufficient compact S_i and usc u_i; in general, things can go awry (see Dufwenberg & Stegeman 2004 Ecta)

Iterated Elimination of Strictly Dominated Strategies (IESDS)

Consider IESDS for game with mixed strategies:

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(iv) No $\boldsymbol{\sigma}_i \in \Sigma_i^\infty$ is strictly dominated in the game $\langle I, \Sigma^\infty, u \rangle$.

Lemma

$\boldsymbol{\sigma}_i \in \Sigma_i^\infty \implies \text{supp}(\boldsymbol{\sigma}_i) \subseteq S_i^\infty$.

Why?

Iterated Elimination of Strictly Dominated Strategies (IESDS)

		Col Player	
		L	R
	T	3,1	0,1
Row Player	M	0,1	3,1
	B	2,1	2,1

No pure strategies are strictly dominated: $S = S^{\infty}$.

Yet, 1/5 T + 1/5 M is strictly dominated by B.

Conclusion: $S_i^{\infty} = \text{supp}(\Sigma_i^{\infty})$ but $\Delta(S_i^{\infty}) \neq \Sigma_i^{\infty}$.

Iterated Elimination of Strictly Dominated Strategies (IESDS)

Definition

$\Gamma = \langle I, S, u \rangle$ is **dominance-solvable** if $|S^\infty| = 1$, i.e., a single strategy profile survives IESDS.

A very strong prediction.

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A very strong prediction.

Corollary

$|S^\infty| = 1 \iff |\Sigma^\infty| = 1$.

Example: Provision of Public Goods

Example

N team members decide how much time to allocate to group work vs. individual work.

The quality of the outcome of the shared task depends on the (geometric) avg. effort/time spent of the team: $\prod_j s_j^{1/N}$.

The quality of the outcome of the individual task only depends on the individual time spent: $1 - s_i$.

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Get paid $\alpha \in (1, N)$ for the quality of the shared task and 1 for the individual task.

Goal: maximise payment.

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Strategy space $S_i := [0, 1]$.

Payoffs: $u_i(s) = \alpha(\prod_j s_j^{1/N}) + 1 - s_i$.

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Claim: there is no strictly dominant strategy.

For any $s_i > 0$, and $\prod_{j \neq i} s_j = 0$, $u_i(s_i, s_{-i}) = 1 - s_i < 1 = u_i(0, s_{-i})$.

Hence, $\forall s_i > 0$, s_i cannot be strictly dominant.

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Hence, $\forall s_i > 0$, s_i cannot be strictly dominant.

Moreover, for $s_i = 0$ and $\prod_{j \neq i} s_j = 1$, $u_i(0, s_{-i}) = 1 < \alpha = u_i(1, s_{-i})$.

Hence, $s_i = 0$ cannot be strictly dominant.

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$S_i := [0,1]$; $\bar{s}_i := \prod_{j \neq i} s_j$ $u_i(s) = \alpha s_i^{1/N} \bar{s}_i^{1/N} + 1 - s_i$.

Claim: the game is dominance solvable.

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Let $s_{(1)} := (\alpha/N)^{N/(N-1)}$. WTS any $s_i > s_{(1)}$ is strictly dominated by $s_{(1)}$. $\forall s_{-i}$.

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& u_i(s_{(1)} + e, s_{-i}) - u_i(s_{(1)}, s_{-i}) \\
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& = \alpha \bar{s}_i^{1/N} \left([s_{(1)} + e]^{1/N} - [s_{(1)}]^{1/N}\right) - e \\
& < \alpha \bar{s}_i^{1/N} \frac{1}{N} [s_{(1)}]^{1/N - 1} e - e
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& = \left(\bar{s}_i^{1/N} \frac{\alpha}{N} [s_{(1)}]^{-(N-1)/N} - 1\right) e \\
& \leq \left(\frac{\alpha}{N} [s_{(1)}]^{-(N-1)/N} - 1\right) e < 0.
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$s_{(k+1)} := s_{(k)}(\alpha/N)^{N/(N-1)}$ given $s_j \in [0, s_{(k)}] \implies \bar{s}_i \in [0, (s_{(k)})^{N-1}]$.

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&= \left(\bar{s}_i^{1/N} \frac{\alpha}{N} [s_{(k+1)}]^{-(N-1)/N} - 1\right) e \\
&\leq \left((s_{(k)})^{(N-1)/N} \frac{\alpha}{N} [s_{(k+1)}]^{-(N-1)/N} - 1\right) e
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&= \alpha \bar{s}_i^{1/N} \left([s_{(k+1)} + e]^{1/N} - [s_{(k+1)}]^{1/N}\right) - e \\
&< \frac{\alpha}{N} \bar{s}_i^{1/N} [s_{(k+1)}]^{1/N-1} e - e \\
&= \left(\bar{s}_i^{1/N} \frac{\alpha}{N} [s_{(k+1)}]^{-(N-1)/N} - 1\right) e \\
&\leq \left((s_{(k)})^{(N-1)/N} \frac{\alpha}{N} [s_{(k+1)}]^{-(N-1)/N} - 1\right) e \\
&= \left((s_{(k)})^{(N-1)/N} [s_{(k)}(\alpha/N)^{N/(N-1)}]^{-(N-1)/N} \frac{\alpha}{N} - 1\right) e \leq 0.
\end{aligned}
$$

Example: Provision of Public Goods

Example

$S_i := [0,1]$; $\bar{s}_i := \prod_{j \neq i} s_j$ $u_i(s) = \alpha s_i^{1/N} \bar{s}_i^{1/N} + 1 - s_i$.

Claim: the game is dominance solvable.

Shown: for any k, any $s_i > s_{(k+1)}$ is iteratedly strictly dominated by $s_{(k+1)} := s_{(k)}(\alpha/N)^{N/(N-1)}$ given $s_j \in [0, s_{(k)}] \implies \bar{s}_i \in [0, (s_{(k)})^{N-1}]$.

With $s_{(0)} := 1$, defines decreasing sequence: $s_{(k)} := s_{(k-1)}(\alpha/N)^{N/(N-1)} = (\alpha/N)^{kN/(N-1)}$ and $\lim_{k \to \infty} s_{(k)} = 0$.

Overview

1. Strategic Interaction
2. Normal-Form Games
3. Strict Dominance
4. Iterated Elimination of Strictly Dominated Strategies (IESDS)
5. Weak Dominance
   - 2nd-Price Auction
6. Rationalisability
7. Level-*k*
8. More

Original Split or Steal (Golden Balls, ITV 2007-09)

		Col Player	
		Split	Steal
Row Player	Split	J/2, J/2	0, J
	Steal	J, 0	0, 0

No strictly dominant strategies.

Prediction?

Weak Dominance

Definition

Fix $\Gamma = \langle I, S, u \rangle$.

(i) Strategy $\sigma_i \in \Sigma_i$ of player i **is weakly dominated by** strategy $\sigma_i' \in \Sigma_i$ iff $u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma_i', \sigma_{-i})\ \forall \sigma_{-i} \in \Sigma_{-i}$ and $\exists \sigma_{-i}' \in \Sigma_{-i} : u_i(\sigma_i, \sigma_{-i}') < u_i(\sigma_i', \sigma_{-i}')$.

(ii) Strategy $\sigma_i \in \Sigma_i$ of player i **weakly dominant** iff it weakly dominates every other strategy σ_i'.

Define weak dominance relation for player i.

Is it reflexive? Complete? Transitive? Does it induce a lattice?

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(iii) s_i is weakly dominant if and only if $\forall s_i' \neq s_i,\ u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),\ \forall s_{-i} \in S_{-i}$ and $\exists s_{-i}' \in S_{-i} : u_i(s_i, s_{-i}') > u_i(s_i', s_{-i}')$.
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(v) If s_i is weakly dominated, then any $\sigma_i \in \Sigma_i : \sigma_i(s_i) > 0$ is weakly dominated.
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(vi) σ_i is weakly dominated $\iff \forall \alpha \in (0,1], \forall \sigma_i' \in \Sigma_i,\ \alpha\sigma_i + (1-\alpha)\sigma_i'$ is weakly dominated. (Bis.)

Iterated Elimination of Weakly Dominated Strategies

		Col Player		
		A2	B2	C2
	A1	2,0	0,0	1,0
Row Player	B1	1,1	1,1	1,1
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What survives IEWDS?

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Conclusion: Order of deletion matters!

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Iterated admissibility: maximal simultaneous deletion of weakly dominated actions

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2nd-Price Auction

I bidders with valuations $0 \leq v_i$ and $v_i \leq v_{i+1}$. Bids $s_i \geq 0$.

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Claim: $s_i = v_i$ is weakly dominant.

(i) $\forall s_i', s_{-i}$: (a) $s_i', v_i > \max_{j \neq i} s_j$, (b) $\max_{j \neq i} s_j > s_i', v_i$, (c) $\max_{j \neq i} s_j = v_i$,
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$s_i = v_i$ and s_i' yield same payoff.

(ii) $\forall s_i', s_{-i} : s_i' \geq \max_{j \neq i} s_j > v_i = s_i$: make a strict loss with s_i' and no loss with $s_i = v_i$.

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Claim: $s_i = v_i$ is weakly dominant.

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$s_i = v_i$ and s_i' yield same payoff.

(ii) $\forall s_i', s_{-i} : s_i' \geq \max_{j \neq i} s_j > v_i = s_i$: make a strict loss with s_i' and no loss with $s_i = v_i$.

(iii) $\forall s_i', s_{-i} : s_i = v_i > s_i' = \max_{j \neq i} s_j$: make strictly more with $s_i = v_i$ (win wp 1, pay same).

2nd-Price Auction

I bidders with valuations $0 \leq v_i$ and $v_i \leq v_{i+1}$. Bids $s_i \geq 0$.

2PA: Highest bid wins and pays 2nd highest bid.

Payoffs:

$$u_i(s_i, s_{-i}) = v_i - \max_{j \neq i} s_j \text{ if } s_i > \max_{j \neq i} s_j.$$
$$u_i(s_i, s_{-i}) = \frac{1}{|j : s_j = s_i|}(v_i - s_i) \text{ if } s_i = \max_{j \neq i} s_j.$$
$$u_i(s_i, s_{-i}) = 0 \text{ if } s_i < \max_{j \neq i} s_j.$$

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(iv) $v_i > \max_{j \neq i} s_j > s_i'$: make zero with s_i'; could make strictly positive payoff with $s_i = v_i$.

Overview

Best Response

Definition

- $\sigma_i \in \Sigma_i$ is a **best response to** $\sigma_{-i} \in \Sigma_{-i}$ iff $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ $\forall \sigma_i' \in \Sigma_i$.

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Important: always need to consider mixed strategies!

Best Response

		Col Player	
		A2	B2
	A1	2,1	0,1
Row Player	B1	1,1	1,1
	C1	0,1	2,1

B1 is BR to σ_2 iff σ_2 is 1/2 A2 + 1/2 B2.

Best Response

		Col Player	
		A2	B2
	A1	3,1	0,1
Row Player	B1	2,1	2,1
	C1	0,1	3,1

Even if all pure strategies in support are BR to something, it does not mean that mixed strategy is.

E.g., 1/2 A1 + 1/2 C1 is never a BR to any strategy of Row.

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Let $s_i^* \in \arg\max_{s_i \in \text{supp}(\boldsymbol{\sigma}_i)} u_i(s_i, \boldsymbol{\sigma}_{-i})$ and $s_i \in \text{supp}(\boldsymbol{\sigma}_i)$ but $u_i(s_i, \boldsymbol{\sigma}_i) < u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i})$. Then,

$$u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \leq \boldsymbol{\sigma}_i(s_i) u_i(s_i, \boldsymbol{\sigma}_{-i}) + (1 - \boldsymbol{\sigma}_i(s_i)) u_i(s_i^*, \boldsymbol{\sigma}_{-i})$$

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As $u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s_i, \boldsymbol{\sigma}_{-i}) \forall s_i \in \mathrm{supp}(\boldsymbol{\sigma}_i)$, then it must be an extreme point of the convex hull (an interval), and so $u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) = \max_{s_i \in \mathrm{supp}(\boldsymbol{\sigma}_i)} u_i(s_i, \boldsymbol{\sigma}_{-i})$.

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a contradiction.

Best Response

Proposition

(i) A strictly dominated strategy is never a best response.

(ii) In finite 2-player games, a pure strategy is never a best-response if and only if it is strictly dominated.

(i) is immediate. (ii) is an application of separating hyperplane theorem.

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Will show you another way of proving (ii).

Proof Sketch for (ii)

Suppose σ_i^* is not strictly dominated. Define $f : \Sigma_i \rightrightarrows \Sigma_j$ s.t. $f(\sigma_i) := \{\sigma_j | u_i(\sigma_i^*, \sigma_j) \geq u_i(\sigma_i, \sigma_j)\}$. Let $b_i(\sigma_j) : \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_j)$. Define $g : \Sigma \rightrightarrows \Sigma$ s.t. $g(\sigma_i, \sigma_j) = b_i(\sigma_j) \times f(\sigma_i)$.

(1) Prove that g is nonempty-valued, convex-valued, compact-valued, and UHC.

(2) Argue that $\exists \sigma \in \Sigma : \sigma \in g(\sigma)$.

(3) Argue that σ_i^* is not a never best response.

(4) Conclude that in finite 2-player games, a pure strategy is never a best-response if and only if it is strictly dominated.

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- $f(\boldsymbol{\sigma}_i)$ compact $\because$ $f(\boldsymbol{\sigma}_i)$ closed and $\subseteq \Sigma_j$ compact.
- f UHC: $\forall (\boldsymbol{\sigma}_i^n, \boldsymbol{\sigma}_j^n)_n : (\boldsymbol{\sigma}_i^n, \boldsymbol{\sigma}_j^n) \to (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)$ and $\boldsymbol{\sigma}_j^n \in f(\boldsymbol{\sigma}_i^n)$, $\mathbf{0} \leq \lim_{n\to\infty} u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j^n) - u_i(\boldsymbol{\sigma}_i^n, \boldsymbol{\sigma}_j^n) = u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j) - u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)$ $\because$ continuity u_i.

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(b) Let $b_i(\sigma_j) : \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_j)$.

- As u_i is continuous and linear in σ_i, and Σ_i compact, b_i is nonempty-valued, compact-valued, convex-valued, and UHC (by Berge's theorem of the maximum).

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(c) Define $g : \Sigma \rightrightarrows \Sigma$ s.t. $g(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = b_i(\boldsymbol{\sigma}_j) \times f(\boldsymbol{\sigma}_i)$.

- f, b_i nonempty-valued, convex-valued, compact-valued, and UHC $\implies$ g too. (Prove it!)

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Proof for (ii)

That strictly dominated implies NBR is immediate. Consider then a strategy that is not strictly dominated, $\boldsymbol{\sigma}_i^*$. WTS it implies that it is a BR to some $\boldsymbol{\sigma}_j^*$.

(a) Define $f : \Sigma_i \rightrightarrows \Sigma_j$ s.t. $f(\boldsymbol{\sigma}_i) := \{\boldsymbol{\sigma}_j | u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)\}$.

(b) Let $b_i(\boldsymbol{\sigma}_j) : \arg\max_{\boldsymbol{\sigma}_i \in \Sigma_i} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)$.

(c) Define $g : \Sigma \rightrightarrows \Sigma$ s.t. $g(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) = b_i(\boldsymbol{\sigma}_j) \times f(\boldsymbol{\sigma}_i)$.

(d) By Kakutani's FPThm, $\exists \boldsymbol{\sigma} \in \Sigma : (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) \in g(\boldsymbol{\sigma}) \implies \boldsymbol{\sigma}_i \in b_i(\boldsymbol{\sigma}_j)$ and $\boldsymbol{\sigma}_j \in f(\boldsymbol{\sigma}_i)$.

- $\boldsymbol{\sigma}_i \in b_i(\boldsymbol{\sigma}_j) \implies u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_j)\ \forall \boldsymbol{\sigma}'_i \in \Sigma_i$.
- $\boldsymbol{\sigma}_j \in f(\boldsymbol{\sigma}_i) \implies u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)$.

Best Response

Proposition

(i) A strictly dominated strategy is never a best response.

(ii) In finite 2-player games, a pure strategy is never a best-response if and only if it is strictly dominated.

Proof for (ii)

That strictly dominated implies NBR is immediate. Consider then a strategy that is not strictly dominated, $\boldsymbol{\sigma}_i^*$. WTS it implies that it is a BR to some $\boldsymbol{\sigma}_j^*$.

(a) Define $f : \Sigma_i \rightrightarrows \Sigma_j$ s.t. $f(\boldsymbol{\sigma}_i) := \{\boldsymbol{\sigma}_j | u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j)\}$.

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(e) Conclude $\boldsymbol{\sigma}_i^*$ BR to $\boldsymbol{\sigma}_j$.
   - $\because u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) \geq u_i(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_j) \ \forall \boldsymbol{\sigma}_i' \in \Sigma_i$.

Rationalisatibility

Definition (Bernheim, 1984 Ecta; Pearce, 1984 Ecta)

Given $\langle I, S, u \rangle$, let $\Sigma_j^0 := \Sigma_j$ for all j.

(i) $\sigma_i \in \Sigma_i$ is k-**rationalisable** for player i if it is a best response to some $\sigma_{-i} \in \times_{j \neq i} \text{co}\left(\Sigma_j^{k-1}\right)$, where Σ_j^{k-1} the set of $(k-1)$−rationalisable strategies for player j.

(ii) $\sigma_i \in \Sigma_i$ is **rationalisable** for player i if it is k-rationalisable for all $k \geq 1$.

Rationalisability as iterated elimination of NBRs.

Why convex hull? Two pure strategies may be BR to some opponents' strategy profile, but mixture between them may not and player may be unsure of which of the surviving strategies to use.

Rationalisatibility

Lemma

σ_i rationalisable only if σ_i survives IESDS.

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Conclude: set of rationalisable strategies is a subset of set of strategies surviving IESDS.

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For any finite game, the set of rationalisable strategy profiles is nonempty.

Proof later.

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Proof later.

Lemma

Any pure strategy in the support of a rationalisable mixed strategy is rationalisable.

Why?

Recall that, if σ_i is BR to σ_{-i}, then so are any $s_i \in \text{supp}(\sigma_i)$.

Rationalisatibility

Definition (Pearce, 1984 Ecta)

Given $\langle I, S, u\rangle$, let $\Sigma_j^0 := \Sigma_j$ for all j.

(i) $\sigma_i \in \Sigma_i$ is k**-rationalisable with correlation** for player i if it is a best response to some $\sigma_{-i} \in \Delta\left(\times_{j \neq i}\Sigma_j^{C,k-1}\right)$, where $\Sigma_j^{C,k-1}$ the set of strategies which are $(k-1)-$rationalisable with correlation for player j.

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If rationalisable without correlation, then rationalisable with correlation?

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If rationalisable without correlation, then rationalisable with correlation?

Is the converse also true?

Rationalisatibility

Proposition 1 (Pearce 1984 Ecta)

Any pure strategy in the support of a mixed strategy which is rationalisable with correlation is rationalisation with correlation.

Again, recall that, if $\boldsymbol{\sigma}_i$ is BR to $\boldsymbol{\sigma}_{-i}$, then so are any $s_i \in \text{supp}(\boldsymbol{\sigma}_i)$.

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Lemma 3 (Pearce 1984 Ecta)

A strategy is 1-rationalisable with correlation if and only if it is not strictly dominated. Furthermore, the set of strategy profiles which are rationalisable with correlation corresponds to the set of strategy profiles surviving IESDS.

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Why?

Proof Intuition

Recall that in finite 2-player games, a pure strategy is never a best-response if and only if it is strictly dominated.
For each player i and k, take $-i$ as player who is choosing in $\Delta(S_{-i}^{C,k-1})$.

Rationalisability

Proposition

$\exists \boldsymbol{\sigma}_{-i} \in \text{int}(\Delta(A_{-i}))$ s.t. $S' \subseteq \arg\max_{s_i \in S_i} u_i(s_i, \boldsymbol{\sigma}_{-i})$ if and only if $\nexists \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_i' \in \Delta(A_i)$: $\text{supp}(\boldsymbol{\sigma}_i') \subseteq S'$ and $\boldsymbol{\sigma}_i$ weakly dominates $\boldsymbol{\sigma}_i$.

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Corollary

$\boldsymbol{\sigma}_i$ is not weakly dominated if and only if it is a best response to some $\boldsymbol{\sigma}_{-i} \in \Delta(A_{-i})$.

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Problem set question.

Second-Price Auction

I bidders with valuations $0 \leq v_i$ and $v_i \leq v_{i+1}$. Bids $s_i \geq 0$.

2PA: Highest bid wins and pays 2nd highest bid.

Payoffs:

$u_i(s_i, s_{-i}) = v_i - \max_{j \neq i} s_j$ if $s_i > \max_{j \neq i} s_j$.

$u_i(s_i, s_{-i}) = \frac{1}{|j : s_j = s_i|}(v_i - s_i)$ if $s_i = \max_{j \neq i} s_j$.

$u_i(s_i, s_{-i}) = 0$ if $s_i < \max_{j \neq i} s_j$.

Claim: Every strategy is rationalisable.

A Game

In a piece of paper, please write any number in [0, 100].

You have 2 minutes to think about it.

You win if you get the closest to 2/3 of the class average.

You should not disclose any information to your colleages.

Overview

Level-k

WT incorporate reasoning mistakes.

Level-k

- Stahl (1993 GEB), Stahl and Wilson (1995 GEB), Nagel (1995 AER)
- Consider dominance-solvable game.
- Fix $\sigma_i^0 \in \Delta(S_i)$.
- A **level-k** player chooses a best response to $k - 1$ level players: $s_i^k = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^{k-1})$.

Level-*k*

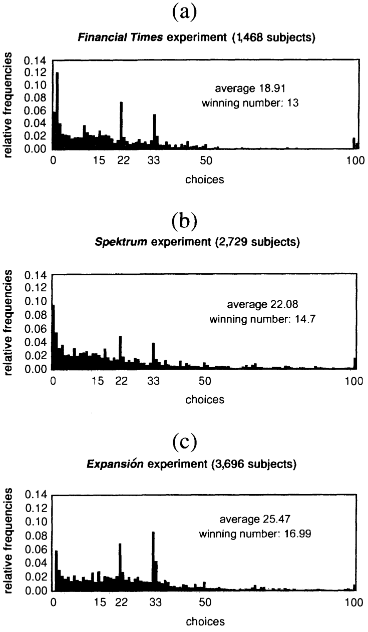
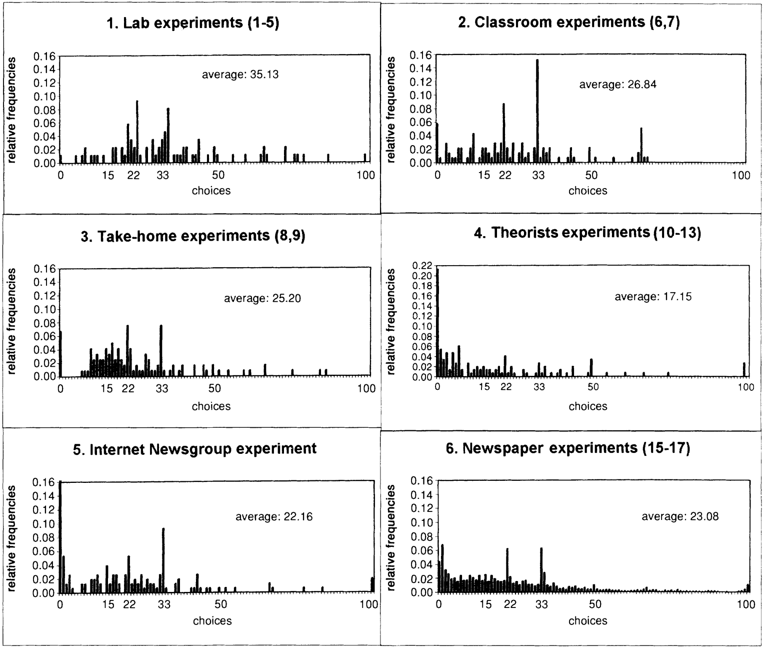


FIGURE 1. RELATIVE FREQUENCIES OF CHOICES

Bosch-Domènech, Montalvo, Nagel, & Satorra (2002 AER). Guess 2/3 of Average.

Peaks around 33 = BR(50), 22 = BR(33), and the dominance solution 0.

Level-*k*


1. Lab experiments (1-5)
average: 35.13
2. Classroom experiments (6,7)
average: 26.84
3. Take-home experiments (8,9)
average: 25.20
4. Theorists experiments (10-13)
average: 17.15
5. Internet Newsgroup experiment
average: 22.16
6. Newspaper experiments (15-17)
average: 23.08
relative frequencies
choices
0
15
22
33
50
100

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Cognitive Hierarchies

- Camerer, Ho, & Chong (2004 QJE)
- Distribution $P \in \Delta(\mathbb{N}_{\mathbf{0}})$ s.t., level-k best-responds to distribution of levels $\ell < k$ given by $P(\ell | \ell < k)$.
- P exogenous; data fitting device.

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Endogenous Depth of Reasoning

- Alaoui & Penta (2016 RES)
- Endogenous level-k, resulting from cost-benefit analysis of 'reasoning further'.
- Level-0 exogenous; non-equilibrium.

Level-k

WT incorporate reasoning mistakes.

Level-k

Cognitive Hierarchies

Endogenous Depth of Reasoning

Issues

(1) as if people have very unrealistic beliefs.

(2) not well defined for arbitrary games.

(3) "level" unstable even across dominance-solvable games.

(4) individual's reasoning seems to depend on payoffs: take "more steps" of IESDS the higher the stakes.

(5) individual's reasoning seems to react to relative incentives smoothly.

Possible ways forward: more later

Overview

1. Strategic Interaction
2. Normal-Form Games
3. Strict Dominance
4. Iterated Elimination of Strictly Dominated Strategies (IESDS)
5. Weak Dominance
6. Rationalisability
7. Level-*k*
8. More

More

- **Miscellanea:**
  - Rationalisability with preferences over lotteries: Weinstein (2016 Ecta)
  - Potential games (a very useful class of games): Monderer & Shapley (1996 GEB)
  - p-Best response: Tercieux (2006 JET)
  - Chess is Dominance-solvable in 2 steps (!) (Ewerhart, 2000 GEB)
- Applications of Level-k: to macro (Farhi & Werning, 2019 AER); to mechanism design (Kneeland, 2022 JET).
- Rationalisability in networks: Lipnowski & Sadler (2019 Ecta)